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Some Properties on Sensitivity, Transitivity and Mixing of Set-Valued Dynamical Systems

Wong, K. S. and Salleh, Z. *

Department of Mathematics, Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, Malaysia

E-mail: zabidin@umt.edu.my
**Corresponding author*

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Abstract

In this paper, we study the dynamical properties of set-valued dynamical systems. Specifically, we focus on the sensitivity, transitivity and mixing of set-valued dynamical systems. Under the setting of set-valued case, we define sensitivity and investigate its properties. We also study the transitivity and mixing of set-valued dynamical systems that have been defined. We show that both transitivity and mixing are invariant under topological conjugacy. We also discuss some implication results on the product set-valued function constructed from two different set-valued functions equipped with various transitivity and mixing conditions.

Keywords: set-valued dynamical systems; sensitive; topologically transitive; topologically mixing; product dynamical systems.

1 Introduction

Dynamical systems are widely used as mathematical models to study numerous applications in various fields ranging from natural to social sciences. For many years, the chaotic behaviour of the systems or so-called chaos remains one of the biggest interests among researchers. Up till now various notions of chaos have been introduced (see [10]). Oprocha and Zhang [16] analyzed the relationships between different notions of chaos on dynamical systems. In general, these notions of chaos are not equivalent but most of them are studied in view of topology. Two common ingredients for the notions of chaos are transitivity and sensitivity of the system.

Normally the dynamics of the system is investigated in the aspect of individual single point. However, there are problems and cases where one is required to investigate the dynamics of the system in the aspect of a collection of points. In recent years, works and research on the dynamics of set-valued dynamical systems can be found. The results from Román-Flores [6] proved that transitivity on set-valued dynamical system implies transitivty on single-valued dynamical system and gave an example to show that the converse is not always true. Metzger et al. [12] proposed topological stability for set-valued maps and extended several results from classical single-valued cases into set-valued cases. As a continuation work from [19], the aim of this paper is to introduce the concept of sensitivity for set-valued dynamical systems and investigate more properties on the transitivity and mixing of set-valued dynamical systems. We define sensitivity under set-valued setting and show that topologically mixing set-valued function implies sensitive. We also prove that a set-valued dynamical system which contain at least one transitive point does not have sensitive property. Besides that, we show that the transitivity and mixing of set-valued function are invariant under topological conjugacy and discuss some implication results on the product set-valued function constructed from two set-valued functions equipped with different transitivity and mixing conditions.

This paper will be organized as follows. In Section 2 we recall some definitions and introduce some notations. In Section 3 we prove some results on the sensitivity of set-valued dynamical systems. In Section 4 we focus on the invariant properties of transitivity and mixing for set-valued function and show some implications of transitivity and mixing on the product set-valued function.

2 Preliminaries

Let (X, d) be a compact metric space and 2^X be the collection of all nonempty closed subsets of X. We call a function $F : X \to 2^X$ as set-valued function. If $A \subseteq X$, we have $F(A) = \{y \in X :$ there is a point $x \in A$ such that $y \in F(x)\}$. The set-valued function F is said to be upper semicontinuous at $x \in X$ if for any open subset V of X containing $F(x)$, there is an open subset U of X containing x such that for every $t \in U$, then $F(t) \subseteq V$. Moreover, F is upper semicontinuous if it is upper semicontinuous at every point of X. Throughout the paper we represent (X, d) by X and assume the set-valued function F is upper semicontinuous unless explicitly stated.

By (0.8) from [14], we know that 2^X is compact. Therefore, it is clear that every element of 2^X is also a nonempty compact subset of X since they are nonempty closed subsets of X. The pair (X, F) is called as set-valued dynamical system. F^0 is denoted as the identity on X and $F^n = F \circ F^{n-1}$ for all integers $n > 0$. We will list out some definitions that have been defined under the set-valued setting.

Definition 2.1 ([2]). *Let $F : X \to 2^X$ be a set-valued function, then the inverse set-valued function $F^{-1} : X \to 2^X$ is defined by $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in X$.*

Definition 2.2 ([17]). *Let (X, F) be a set-valued dynamical system. For $x \in X$, an orbit of x is a sequence $(x_i)_{i=0}^{\infty}$ such that $x_0 = x$ and $x_{i+1} \in F(x_i)$ for all integers $i \geq 0$. The collection of all orbits of x is called as the complete orbit of x, denoted by $CO(x)$.*

It is obvious that the orbits are not uniquely determined in set-valued case as shown by Example 2.3 in [19]. Note that the product set-valued function $F \times F : X \times X \to 2^{X \times X}$ is defined by $(F \times F)(x, x') = \{(y, y') \in X \times X : y \in F(x) \text{ and } y' \in F(x')\}$ for all $x, x' \in X$.

Definition 2.3 ([19]). *Let (X, F) be a set-valued dynamical system. A set-valued function F is topologically transitive if for any nonempty open subsets U and V of X , there exists $m \in \mathbb{N}$ and $x \in U$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_m \in V$.*

Definition 2.4 ([17]). *A set-valued function F is topologically mixing if for any nonempty open subsets U and V in X, there is a $M \in \mathbb{N}$ such that for any $m > M$ there is an $x \in U$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_m \in V$.*

Definition 2.5 ([19]). *Let F be a set-valued function of a compact metric space X. Then F is said to be*

i) Topologically bitransitive if F^2 is topologically transitive.

ii) Totally transitive if F^n is topologically transitive for all $n \in \mathbb{N}$.

iii) Weakly mixing if the product set-valued function $F \times F$ is topologically transitive.

Recall that in single-valued case, a continuous function $f : X \to X$ is said to be sensitive dependence on initial conditions (or just sensitive) if there exists $\varepsilon > 0$ such that for each $x \in X$ and each $\delta > 0$ there is $y \in X$ with $d(x, y) < \delta$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \varepsilon$. We will generalize this notion to set-valued setting as follows.

Definition 2.6. *Let X be a compact metric space and $F : X \to 2^X$ be a set-valued function on X. F is said to be sensitive dependence on initial conditions if there exists an $\varepsilon > 0$ such that for all points $x \in X$ and for each $\delta > 0$, there exists at least one $y \in X$ with $d(x, y) < \delta$ and a positive integer k such that both x and y have an orbit $(x_i)_{i=0}^{\infty}$ and $(y_i)_{i=0}^{\infty}$ respectively where $d(x_k, y_k) \geq \varepsilon$. The constant value ε is called as sensitive constant.*

While the notion of sensitivity is used to describe the chaotic behaviour of the systems, we use the notion of equicontinuity to describe the stability of the systems. One can see that both notions are in opposition to each other. It is well known that equicontinuous single-valued dynamical systems will only have simple dynamical behaviours and this result is true for set-valued case as well ([15]). Below we define equicontinuous under set-valued setting which extend from the corresponding definition in the single-valued case ([4]).

Definition 2.7. *Let (X, F) be a set-valued dynamical system. The set-valued function F is said to be equicontinuous at $x \in X$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $y \in X$ with $d(x, y) < \delta$, both x and y have an orbit $(x_i)_{i=0}^{\infty}$ and $(y_i)_{i=0}^{\infty}$ respectively satisfy $d(x_k, y_k) < \varepsilon$ for all positive integers k. The point $x \in X$ that satisfy the condition is known as equicontinuous point. Moreover, F is said to be equicontinuous if every point $x \in X$ is an equicontinuous point.*

Another useful concept in dynamical systems is topological conjugacy which use to describe the similarities in dynamical behaviour between two different systems. Topological conjugacy allows one to investigate the dynamical behaviour of some new or complicated dynamical systems by comparing it with a well-known dynamical system. Below is the definition of topological conjugacy under set-valued setting.

Definition 2.8 ([9])**.** *Let (X, F) and (Y, G) be set-valued dynamical systems. G is topologically semi-conjugate to F if there exists a continuous surjection $h : X \to Y$ such that $(G \circ h)(x) \subseteq (h \circ F)(x)$ for all $x \in X$. The surjection h is called as topological semi-conjugacy from (X, F) to (Y, G). We say that G is topologically conjugate to F if there exists a homeomorphism function $h : X \to Y$ such that $(G \circ h)(x) = (h \circ F)(x)$ for all $x \in X$. The function h is called as topological conjugacy between (X, F) and (Y, G).*

3 Sensitivity of Set-valued Function

Sensitive is one of the main components in various notions of chaos, for e.g. Li-Yorke Sensitivity, Devaney Chaos, and Auslander-Yorke Chaos. Akin and Kolyada [1] proposed Li-Yorke sensitivity and proved that weak mixing systems are Li-Yorke sensitive. Devaney [5] studied his own version of chaos infused with concept of sensitivity in various dimensional dynamics. Auslander and Yorke [3] gave a new notion of chaos and proved that any map on an interval satisfying a generalized period three condition have chaos. It is well known that both sensitivity and transitivity are closely related in single-valued dynamical systems (see [10]). We will show that in set-valued case a topologically mixing set-valued function will implies sensitive.

Theorem 3.1. *Let (X, F) be a set-valued dynamical system with X has at least two points. If F is topologically mixing, then F has sensitive dependence on initial conditions.*

Proof. Let $u, v \in X$ such that $u \neq v$ and let $\varepsilon = d(u, v)/4$. We show that ε is the constant that satisfy the condition for sensitivity. Denote $B_\varepsilon(u)$ as the open ball center at u with radius ε and $B_\varepsilon(v)$ as the open ball center at v with radius ε. Similarly, for any $x \in X$ and any $\delta > 0$ we denote $B_\delta(x)$ as the open ball of x with radius δ. Since F is topologically mixing, there exists $M_1 \in \mathbb{N}$ such that for any $m > M_1$ there is $y \in B_\delta(x)$ with an orbit $(y_i)_{i=0}^{\infty}$ such that $y_m \in B_\varepsilon(u)$ and there exists $M_2 \in \mathbb{N}$ such that for any $m > M_2$ there is $z \in B_\delta(x)$ with an orbit $(z_i)_{i=0}^{\infty}$ such that $z_m \in B_\varepsilon(v)$. Let $M = \max\{M_1, M_2\}$ so that the orbit $(y_i)_{i=0}^{\infty}$ and $(z_i)_{i=0}^{\infty}$ satisfy $y_m \in B_\varepsilon(u)$ and $z_m \in B_\varepsilon(v)$ for all $m > M$. By generalized triangle inequality,

$$d(u, v) \leq d(u, y_m) + d(y_m, z_m) + d(z_m, v).$$

Since $d(u, v) = 4\varepsilon$, $d(u, y_m) < \varepsilon$ and $d(z_m, v) < \varepsilon$, we obtain

$$2\varepsilon \leq d(y_m, z_m) \leq d(y_m, x_m) + d(x_m, z_m).$$

Let $(x_i)_{i=0}^{\infty}$ be any orbit of x, then for some $m > M$ it follows that either $d(x_m, y_m) \geq \varepsilon$ or $d(x_m, z_m) \geq \varepsilon$. Therefore, F is sensitive dependence on initial conditions. □

Recall that a set-valued function F is topologically transitive if and only if there is a point $x \in X$ with an orbit $(x_i)_{i=0}^{\infty} \in CO(x)$ such that the orbit is dense in X (see Theorem 3.2 in [19]). Point that satisfies such condition is called as transitive point. We give the definition of almost equicontinuous and show that the set of transitive points is equivalent to the set of equicontinuous points for a topologically transitive set-valued function.

Definition 3.1. *Let X be a compact metric space and F be a topologically transitive set-valued function. If there exists at least one transitive point which is also an equicontinuous point, then F is said to be almost equicontinuous.*

Lemma 3.1. *Let (X, F) be a set-valued dynamical system with F is a topologically transitive set-valued function. If the point $x \in X$ is an equicontinuous point, then x is also a transitive point.*

Proof. Let y be an arbitrary point of X and $\varepsilon > 0$. Since x is equicontinuous point, there exists a $\delta > 0$ such that for all $z \in X$ satisfy $d(x, z) < \delta$, then both x and z have an orbit $(x_i)_{i=0}^{\infty}$ and $(z_i)_{i=0}^{\infty}$ respectively such that $d(x_n, z_n) < \varepsilon/2$ for all positive integers n. Since F is topologically transitive, for the open ball $B_{\varepsilon/2}(y)$ and $B_\delta(x)$, we can find $m \in \mathbb{N}$ and $z \in B_\delta(x)$ with an orbit $(z_i)_{i=0}^{\infty}$ such that $z_m \in B_{\varepsilon/2}(y)$. Then, by using triangle inequality, we have

$$d(x_m, y) \leq d(x_m, z_m) + d(z_m, y) < \varepsilon/2 + \varepsilon/2 < \varepsilon,$$

which implies that $x_m \in B_\varepsilon(y)$. Since y is chosen arbitrary, this means that x has an orbit that is dense in X. Therefore, we conclude that x is a transitive point. □

It is shown in [8] that for single-valued dynamical system (X, f) where f is a self-continuous mapping of a nonempty set X, if f is topologically transitive, then either f is sensitive or f is almost equicontinuous. The theorem below shows the connection between the equicontinuity and sensitivity of a topologically transitive set-valued function.

Theorem 3.2. *Let (X, F) be a set-valued dynamical system with F is a topologically transitive set-valued function. F is almost equicontinuous if and only if F is not sensitive dependence on initial conditions.*

Proof. It is clear that if F is almost equicontinuous then F is not sensitive dependence on initial conditions.

Now for the converse, suppose that F is not sensitive dependence on initial conditions. This means that for all $\varepsilon > 0$ there exists $x \in X$ and $\delta > 0$ such that for all $y \in X$ with $d(x, y) < \delta$, both x and y have an orbit $(x_i)_{i=0}^{\infty}$ and $(y_i)_{i=0}^{\infty}$ respectively satisfy $d(x_k, y_k) < \varepsilon$ for all positive integer k. Clearly, we can see that such point x satisfy the condition of being an equicontinuous point. By Lemma 3.1 this implies that x is also a transitive point. Therefore F is almost equicontinuous. □

There are other notions of sensitivity with different level of strengths such as collectively sensitivity [18], generalized sensitivity [20], syndetically sensitivity and cofinitely sensitivity [13]. Among these notions, collectively sensitivity is one of the stronger forms where the sensitivity is defined in view of a collection of points. We extend this notion to set-valued case and show its relation with topologically weakly mixing set-valued function.

Definition 3.2. *Let X be a compact metric space and $F : X \to 2^X$ be a set-valued function on X. F is said to be collectively sensitive with the collective sensitive constant ε if for any finite many distinct points $x_1, x_2, \ldots, x_n$ of X and any $\delta > 0$, there exist distinct points $y_1, y_2, \ldots, y_n$ of X and a positive integer k such that x_i and y_i has an orbit $\left(x_i^{(j)}\right)_{j=0}^{\infty}$ and $\left(y_i^{(j)}\right)_{j=0}^{\infty}$ respectively for all $i \in \{1, 2, \ldots, n\}$ that satisfy the following conditions:*

i) $d(x_i, y_i) < \delta$ *for all* $i \in \{1, 2, \ldots, n\}$.

ii) *There exists an* $i_0 \in \{1, 2, \ldots, n\}$ *such that* $d\left(x_i^{(k)}, y_{i_0}^{(k)}\right) \geq \varepsilon$ *or* $d\left(x_{i_0}^{(k)},\ y_i^{(k)}\right) \geq \varepsilon$ *for all* $i \in \{1, 2, \ldots, n\}$.

We provide the following lemma here for the sake of completeness.

Lemma 3.2 ([19])**.** *Let* (X, F) *be a set-valued dynamical system and the iterations of any open set is an open set of* X. *If the set-valued function* F *is topologically weakly mixing, then the product set-valued dynamical system* $(X^n, F \times \cdots \times F)$ *is topologically transitive for all integers* $n \geq 1$.

Theorem 3.3. *Let* (X, F) *be a set-valued dynamical system and the iterations of any open set is an open set of* X. *If* F *is topologically weakly mixing, then* F *is collectively sensitive.*

Proof. Let $s_1, s_2 \in X$ be two distinct points with $d(s_1, s_2) \geq 4\varepsilon$ and $G_j = \{x \in X : d(x, s_j) < \varepsilon\}$ for $j = 1, 2$. Clearly G_1 and G_2 are two disjoint nonempty open subsets of X. We will show that F is collectively sensitive with the constant ε. Let $x_1, x_2, \ldots, x_n$ be n distinct points of X and $\delta > 0$. For all $i \in \{1, 2, \ldots, n\}$, denote $B(x_i, \delta)$ as the open ball centered at x_i with radius δ. Since F is topologically weakly mixing, by Lemma 3.2 any k-product set-valued function of F is topologically transitive. Choose $k = 2n$, then for each open sets $B(x_i, \delta)$ and G_j where $i \in \{1, 2, \ldots, n\}$ and $j = 1, 2$, there exists $m \in \mathbb{N}$ and $z_i, w_i \in B(x_i, \delta)$ with an orbit $\left(z_i^{(k)}\right)_{k=0}^{\infty}$ and $\left(w_i^{(k)}\right)_{k=0}^{\infty}$ respectively such that $z_i^{(m)} \in G_1$ and $w_i^{(m)} \in G_2$. By generalized triangle inequality,

$$d(s_1, s_2) \leq d\left(s_1, z_i^{(m)}\right) + d\left(z_i^{(m)}, w_i^{(m)}\right) + d\left(w_i^{(m)}, s_2\right),$$

and this implies that $d\left(z_i^{(m)}, w_i^{(m)}\right) \geq 2\varepsilon$ for all $i \in \{1, 2, \ldots, n\}$. Hence, each x_i will have at least an orbit $\left(x_i^{(j)}\right)_{j=0}^{\infty}$ such that $x_i^{(m)}$ will satisfy

$$2\varepsilon \leq d\left(z_i^{(m)}, w_i^{(m)}\right) \leq d\left(z_i^{(m)}, x_i^{(m)}\right) + d\left(x_i^{(m)}, w_i^{(m)}\right).$$

This means that either $d\left(x_i^{(m)}, z_i^{(m)}\right) \geq \varepsilon$ or $d\left(x_i^{(m)}, w_i^{(m)}\right) \geq \varepsilon$. If $d\left(x_i^{(m)}, z_i^{(m)}\right) \geq \varepsilon$, let $y_i = z_i$. If $d\left(x_i^{(m)}, w_i^{(m)}\right) \geq \varepsilon$, let $y_i = w_i$. Therefore, there exists $y_i \in B(x_i, \delta)$ such that $d\left(x_i^{(m)}, y_i^{(m)}\right) \geq \varepsilon$ for all $i \in \{1, 2, \ldots, n\}$ and this shows that F is collectively sensitive with the collective sensitively constant ε. □

4 More Properties on Transitivity and Mixing of Set-valued Function

In single-valued dynamical systems, both topologically transitive and mixing are preserved by topological conjugacy (see [11]). We will show that these results remain true for set-valued case. In addition, we show that topologically weakly mixing is invariant under conjugacy as well.

Theorem 4.1. *Let* (X, F) *and* (Y, G) *be set-valued dynamical systems where* G *is topologically conjugate with* F. *Then* F *is topologically mixing on* X *if and only if* G *is topologically mixing on* Y.

Proof. Assume F is topologically mixing. Since G is topologically conjugate with F, there exists a homeomorphism function $h : X \to Y$ such that $G \circ h = h \circ F$. Let V_1, V_2 be any two nonempty open subsets of Y. It is clear that $U_1 = h^{-1}(V_1)$ and $U_2 = h^{-1}(V_2)$ are nonempty open subsets of X. Since F is topologically mixing, there exists $M \in \mathbb{N}$ such that for any $m > M$ there is $x \in U_1$ with an orbit $(x_i)_{i=0}^{\infty}$ under F such that $x_m \in U_2$. We will use this orbit to construct a point in V_1 with an orbit such that the iterate of the point greater than M are contained in V_2.

Note that by the definition of orbit for set-valued case,the orbit $(x_i)_{i=0}^{\infty}$ satisfy $x = x_0$ and $x_{i+1} \in F(x_i)$ for all $i \geq 0$. For this orbit of x map each iterate under h and this will lead to $h(x_{i+1}) \in h(F(x_i)) = G(h(x_i))$. Hence, we obtain a point $h(x)$ which is contained in Y and one of its orbit $(h(x_i)_i)_{i=0}^{\infty}$ under G. Since $x \in U_1$, $h(x) \in h(U_1) = V_1$ and $h(x_m) \in h(U_2) = V_2$. This means that there is a point $h(x) \in V_1$ which has an orbit $(h(x_i)_i)_{i=0}^{\infty}$ such that $h(x_m) \in V_2$ for all $m > M$. Since V_1 and V_2 are chosen arbitrary, we conclude that G is topologically mixing.

The converse can be proved exactly with the same arguments by interchanging the role of F and G. □

The next theorem shows the invariant property of topological transitivity for set-valued function.

Theorem 4.2. *Let (X, F) and (Y, G) be set-valued dynamical systems where G is topologically conjugate with F. Then F is topologically transitive on X if and only if G is topologically transitive on Y.*

Proof. Assume F is topologically transitive. Since G is topologically conjugate with F, there exists a homeomorphism function $h : X \to Y$ such that $G \circ h = h \circ F$. Let V_1, V_2 be arbitrary two nonempty open subsets of Y. It is clear that $U_1 = h^{-1}(V_1)$ and $U_2 = h^{-1}(V_2)$ are nonempty open subsets of X by the continuity of h^{-1}. Since F is topologically transitive, there exists $M \in \mathbb{N}$ and $x \in U_1$ with an orbit $(x_i)_{i=0}^{\infty}$ under F such that $x_M \in U_2$.

Note that by the definition of orbit for set-valued case, the orbit $(x_i)_{i=0}^{\infty}$ satisfy $x = x_0$ and $x_{i+1} \in F(x_i)$ for all $i \geq 0$. For this orbit of x, let us map each iterate under h and we have $h(x_{i+1}) \in h(F(x_i)) = G(h(x_i))$ for all $i \geq 0$. Hence, we obtain a point $h(x)$ which is contained in Y and one of its orbit $(h(x_i))_{i=0}^{\infty}$ under G. Since $x \in U_1$ and $x_M \in U_2$, $h(x) \in h(U_1) = V_1$ and $h(x_M) \in h(U_2) = V_2$. This means that there is a point $h(x) \in V_1$ which has an orbit $(h(x_i))_{i=0}^{\infty}$ such that $h(x_M) \in V_2$. Since V_1 and V_2 are chosen arbitrary, we conclude that G is topologically transitive.

The converse can be proved with the similar arguments by interchanging the role of F and G. □

Remark 4.1. *If the function h is topological semi-conjugacy from (X, F) and (Y, G) instead of topological conjugacy, then F topologically transitive on X will implies G topologically transitive on Y. The converse is not true in this case.*

Theorem 4.3. *Let (X, F) and (Y, G) be set-valued dynamical systems where G is topologically conjugate with F. Then, F is topologically weakly mixing on X if and only if G is topologically weakly mixing on Y.*

Proof. Since G is topologically conjugate with F, there is a homeomorphism continuous function $h : X \to Y$ such that $G \circ h = h \circ F$. Let U, V be two nonempty open subsets of $Y \times Y$. Then there exist nonempty open subsets U_1, U_2, V_1, V_2 of Y such that $U_1 \times U_2 \subseteq U$ and $V_1 \times V_2 \subseteq V$. By the homeomorphism of h we have $h^{-1}(U_1), h^{-1}(U_2), h^{-1}(V_1)$ and $h^{-1}(V_2)$ as nonempty open subsets of X. Since F is topologically weakly mixing, there exists $m \in \mathbb{N}$ and $(x, y) \in h^{-1}(U_1) \times h^{-1}(U_2)$ with an orbit $(x_i, y_i)_{i=0}^{\infty}$ such that $(x_m, y_m) \in h^{-1}(V_1) \times h^{-1}(V_2)$. This means that we have $x \in$

$h^{-1}(U_1)$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_m \in h^{-1}(V_1)$ and $y \in h^{-1}(U_2)$ with an orbit $(y_i)_{i=0}^{\infty}$ such that $y_m \in h^{-1}(V_2)$.

By the definition of orbit under set-valued function, the orbit $(x_i)_{i=0}^{\infty}$ satisfy $x_0 = x$ and $x_{i+1} \in F(x_i)$ for all $i \geq 0$. Mapping each points from the orbit under h results $h(x) \in h(h^{-1}(U_1)) = U_1$ and an orbit $h(x_{i+1}) \in h(f(x_i)) = G(h(x_i))$ for all $i \geq 0$. Therefore, we have $h(x) \in U_1$ with an orbit $(h(x_i))_{i=0}^{\infty}$ such that $h(x_m) \in V_1$. Similarly, we have $h(y) \in U_2$ with an orbit $(h(y_i))_{i=0}^{\infty}$ such that $h(y_m) \in V_2$. Hence, we have a point $(h(x), h(y)) \in U_1 \times U_2$ with an orbit $(h(x_i), h(y_i))_{i=0}^{\infty}$ such that $h(x_m, y_m) \in V_1 \times V_2$. This shows that $G \times G$ is topologically transitive which implies that G is topologically weakly mixing.

The converse can be proved in a similar fashion by switching the role of F and G. □

For single-valued case, [7] showed that a dynamical system with topologically mixing property and its direct product with an arbitrary minimal dynamical system has transitive property. Below we present some implications on the product set-valued dynamical systems formed by two set-valued dynamical systems with various transitivity and mixing conditions.

Theorem 4.4. *Let (X, F) and (Y, G) be set-valued dynamical systems. Assume that G is topologically mixing. If F is topologically mixing, then $F \times G$ is topologically mixing.*

Proof. Let U_1, U_2 be nonempty open subsets of X and V_1, V_2 be nonempty open subsets of Y. Since both set-valued functions F and G are topologically mixing, there exists $M_1 \in \mathbb{N}$ such that for any $m > M_1$ there is $x \in U_1$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_m \in U_2$ and there exists $M_2 \in \mathbb{N}$ such that for any $m > M_2$ there is $y \in V_1$ with an orbit $(y_i)_{i=0}^{\infty}$ such that $y_m \in V_2$. Now denote $M = \max\{M_1, M_2\}$, then for all $m > M$ there is $x \in U_1$ with an orbit $(x_i)_{i=0}^{\infty}$ and $y \in V_1$ with an orbit $(y_i)_{i=0}^{\infty}$ such that $x_m \in U_2$ and $y_m \in V_2$. For the product set-valued dynamical system $(X \times Y, F \times G)$, both $U_1 \times V_1$ and $U_2 \times V_2$ are nonempty open subsets of $X \times Y$. Therefore, we have $M \in \mathbb{N}$ such that for all $m > M$ there exists some $(x, y) \in U_1 \times V_1$ with an orbit $(x_i, y_i)_{i=0}^{\infty}$ such that $(x_m, y_m) \in U_2 \times V_2$. This shows that $F \times G$ is topologically mixing. □

From Theorem 4.4, we obtain the following corollary.

Corollary 4.1. *Let (X, F) be a set-valued dynamical system. If F is topologically mixing, then the product set-valued function $F \times F$ is topologically mixing.*

Theorem 4.5. *Let (X, F) and (Y, G) be set-valued dynamical systems. Assume that G is topologically mixing. If F is topologically weakly mixing, then $F \times G$ is topologically weakly mixing.*

Proof. Let U_1, U_2, V_1, V_2 be nonempty open subsets of X. Suppose that F is topologically weakly mixing, this means that the product set-valued function $F \times F$ is topologically transitive. Hence there exists $n \in \mathbb{N}$ and $(x_1, x_2) \in U_1 \times V_1$ with an orbit $\left(x_1^{(i)}, x_2^{(i)}\right)_{i=0}^{\infty}$ such that $\left(x_1^{(n)}, x_2^{(n)}\right) \in U_2 \times V_2$. Since G is topologically mixing, by Corollary 4.1 , $G \times G$ is topologically mixing. This means that for any nonempty open subsets A_1, A_2, B_1, B_2 of Y , we may choose a positive integer $M \leq n$ such that for any $m > M$ there is $(y_1, y_2) \in A_1 \times B_1$ with an orbit $\left(y_1^{(i)}, y_2^{(i)}\right)_{i=0}^{\infty}$ satisfy $\left(x_1^{(n)}, x_2^{(n)}\right) \in A_2 \times B_2$. For product set-valued dynamical system $(X \times Y, F \times G)$, the subsets $U_1 \times A_1, U_2 \times A_2, V_1 \times B_1$ and $V_2 \times B_2$ are nonempty and open in $X \times Y$. Hence, for $(X \times Y) \times (X \times Y)$, we have a point $(x_1, y_1, x_2, y_2) \in (U_1 \times A_1) \times (V_1 \times B_1)$ with an orbit $\left(x_1^{(i)}, y_1^{(i)}, x_2^{(i)}, y_2^{(i)}\right)_{i=0}^{\infty}$ such that $\left(x_1^{(n)}, y_1^{(n)}, x_2^{(n)}, y_2^{(n)}\right) \in (U_2 \times A_2) \times (V_2 \times B_2)$. This shows that $(F \times G) \times (F \times G)$ is topologically transitive which implies that $F \times G$ is topologically weakly mixing. □

Theorem 4.6. *Let (X, F) and (Y, G) be set-valued dynamical systems. Assume that G is topologically mixing. If F is topologically transitive, then $F \times G$ is topologically transitive.*

Proof. Let U_1, U_2 be arbitrary nonempty open subsets of X and V_1, V_2 be arbitrary nonempty open subsets of Y. Suppose that F is topologically transitive, there exists $N \in \mathbb{N}$ and $x \in U_1$ with an orbit $(x_i)_{i=0}^{\infty}$ such that $x_N \in U_2$. Since G is topologically mixing, we can choose a positive integer $M \leq N$ such that for any $m > M$ there is $y \in V_1$ with an orbit $(y_i)_{i=0}^{\infty}$ such that $y_m \in V_2$. As $U_1 \times V_1$ and $U_2 \times V_2$ are nonempty open subsets of $X \times Y$, we have N and a point $(x, y) \in U_1 \times V_1$ with an orbit $(x_i, y_i)_{i=0}^{\infty}$ such that $(x_N, y_N) \in U_2 \times V_2$. Therefore, $F \times G$ is topologically transitive. □

To end this section, we give an equivalent condition of topologically weakly mixing.

Theorem 4.7. *Let (X, F) be a set-valued dynamical system. F is topologically weakly mixing if and only if for any open subsets U, V_1, V_2 of X, there exist $m \in \mathbb{N}$ and $x, y \in U$ with an orbit $(x_i)_{i=0}^{\infty}$ and $(y_i)_{i=0}^{\infty}$ respectively such that $x_m \in V_1$ and $y_m \in V_2$.*

Proof. It is clear that if F is topologically weakly mixing then the stated condition is satisfied. Conversely, suppose that the condition holds and let U_1, U_2, V_1, V_2 be arbitrary nonempty open subsets of X. In order to show that F is topologically weakly mixing, we need to find an $m \in \mathbb{N}$ and $(x, y) \in U_1 \times U_2$ with an orbit $(x_i, y_i)_{i=0}^{\infty}$ such that $(x_m, y_m) \in V_1 \times V_2$. For U_1, U_2 and V_1, by the given condition there exists $n \in \mathbb{N}$ and $x, z \in U_1$ with an orbit $(x_i)_{i=0}^{\infty}$ and $(z_i)_{i=0}^{\infty}$ respectively such that $x_n \in U_2$ and $z_n \in V_1$. Denote

$$U = \{x \in X : x \in U_1 \text{ with an orbit } (x_i)_{i=0}^{\infty} \text{ such that } x_n \in U_2\}.$$

For $\text{int}(U), V_1$ and $\text{int}(F^{-n}(V_2))$, again by the condition there exists $m \in \mathbb{N}$ and $x, w \in \text{int}(U)$ with an orbit $(x_i)_{i=0}^{\infty}$ and $(w_i)_{i=0}^{\infty}$ such that $x_m \in V_1$ and $w_m \in \text{int}(F^{-n}(V_2))$. Since $x \in \text{int}(U) \subset U$, this means that $x \in U_1$ and has an orbit $(x_i)_{i=0}^{\infty}$ such that $x_m \in V_1$. Since $w \in \text{int}(U) \subset U$, $winU_1$ and has an orbit $(w_i)_{i=0}^{\infty}$ such that $w_n \in U_2$. Furthermore, $w_m \in \text{int}(F^{-n}(V_2)) \subseteq F^{-n}(V_2)$ and this implies that $w_{m+n} \in F^n F^{-n}(V_2) \subset V_2 \cap F^n(X)$. Therefore we have a point $w \in U$ with an orbit $(w_i)_{i=0}^{\infty}$ such that $w_n \in U_2$ and $w_{m+n} \in V_2$. Let $y = w_n$ and we complete the proof. □

5 Conclusions

In this paper, we have generalized the notion of sensitivity to set-valued dynamical system and showed that a topologically transitive set-valued function implies sensitive. We also introduced collectively sensitive, which is a stronger form of sensitive and showed that topologically weakly mixing set-valued function implies collectively sensitive. Furthermore, we have proved that both transitivity and mixing properties of set-valued dynamical systems are preserved under topological conjugacy, which similar to their corresponding results in single-valued case. Besides that, we constructed a product set-valued dynamical system from two set-valued dynamical systems with different transitivity and mixing conditions and investigated their effects on the product set-valued dynamical system.

We end this paper with the following questions:

Question 5.1. *Does a set-valued function $F : X \to 2^X$ that is topologically transitive implies sensitive dependence on initial conditions?*

Question 5.2. *For two set-valued dynamical systems (X, F) and (Y, G), what will happen to the product set-valued function $F \times G$ if both G and F are topologically weakly mixing?*

Question 5.3. *For two set-valued dynamical systems (X, F) and (Y, G), what will happen to the product set-valued function $F \times G$ if both G and F are topologically transitive?*

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